COMBINATORICA 2(3) (1982) 213—222

DIPERFECT GRAPHS

Claude BERGE

Dedicated to Tibor Gallai on his seventieth birthday

Received 29 January 1982

Gallai and Milgram have shown that the vertices of a directed graph, with stability number $\alpha(G)$, can be covered by exactly $\alpha(G)$ disjoint paths. However, the various proofs of this result do not imply the existence of a maximum stable set S and of a partition of the vertex-set into paths $\mu_1, \mu_2, \dots, \mu_k$ such tht $|\mu_i \cap S| = 1$ for all i.

Later, Gallai proved that in a directed graph, the maximum number of vertices in a path is at least equal to the chromatic number; here again, we do not know if there exists an optimal coloring $(S_1, S_2, \dots, S_k)$ and a path μ such that $|\mu \cap S_i| = 1$ for all i.

In this paper we show that many directed graphs, like the perfect graphs, have stronger properties: for every maximal stable set S there exists a partition of the vertex set into paths which meet the stable set in only one point. Also: for every optimal coloring there exists a path which meets each color class in only one point. This suggests several conjectuies similar to the perfect graph conjecture.

1. Path-partitions

Let G be a (directed) graph defined by a set X of vertices and a set $U \subset X \times X$ of arcs (directed edges). A *path* $\vec{\mu}$ will be defined as a sequence $(x_1, x_2, \dots, x_k)$ of distinct vertices, such that $(x_1, x_2) \in U, \dots, (x_{k-1}, x_k) \in U$. Every path $\vec{\mu} = (x_1, x_2, \dots, x_k)$ defines a set that we denote by $\mu = \{x_1, x_2, \dots, x_k\}$.

A family $M = \{\mu_1, \mu_2, \dots\}$ is a path-*partition* of G if the μ_i are pairwise vertex-disjoint, and $\bigcup \mu_i = X$.

Denote by $\alpha(G)$ the *stability number* of G, that is the maximum number of independent vertices in G. The purpose of this paper is to enlarge the field of application of the following classical result [7].

Theorem of Gallai—Milgram: Every directed graph G satisfies $\min |M| \leqq \alpha(G)$, *where the minimum is taken over all path-partitions M.*

In other words, one can always cover the vertex-set with exactly $\alpha(G)$ paths which are pairwise disjoint.

In this section we formulate another result which combines an idea of Las Vergnas [8] and an idea of Linial [10]. The proof is essentially the same as in the Gallai—Milgram theorem, but we obtain as corollaries results which could not be obtained directly.

AMS subject classification (1980): 05 C 20; 05 C 15.

Let H be an arborescence; the *root* of H is the vertex x with $d_H^-(x)=0$, and a *sink* of H is a vertex y with $d_H^+(y)=0$. If y is a sink, the maximal path of H leading to y which do not contain a vertex z with $d_H^+(z)\geqq 2$, is called a *terminal branch* of H. Let H be an *arborescence forest* of G, that is a partial graph of G whose connected components are arborescences. We denote by $R(H)$ the set of the roots of these arborescences, and by $S(H)$ the set of the sinks of these arborescences. So a vertex in $R(H)\cap S(H)$ is an isolated vertex of H.

Theorem 1. *Let H_0 be an arborescence forest of G with $R(H_0)=R_0$ and $S(H_0)=S_0$. For every arborescence forest H with $R(H)\subseteqq R_0$, $S(H)\subseteqq S_0$ and $|S(H)|$ minimum, there exists a stable set which meets every terminal branch of H.*

Proof. We assume that the result holds true for all graphs of order less than n, and we consider a graph G of order n. Let H be an arborescence forest of G with $R(H)\subseteqq R_0$, $S(H)\subseteqq S_0$, $|S(H)|$ minimum. If $S(H)$ is stable, the theorem is proved. If $S(H)$ is not stable, there exists in G an arc (b, a) connecting two vertices a and b in $S(H)$.

We have $a\notin R(H)$, because otherwise, $H'=H+(b,a)$ satisfies $R(H')\subseteqq R_0$, $S(H')=S(H)-\{b\}\subseteqq S_0$, $|S(H')|<|S(H)|$, a contradiction. So H has an arc incident to a, say (a_1, a).

Furthermore, $d_H^+(a_1)=1$, because otherwise a_1 has at least two descendents in H which belong to $S(H)$, and $H'=H-(a_1,a)+(b,a)$ satisfies $R(H')\subseteqq R_0$, $S(H')\subseteqq S_0$, $|S(H')|<|S(H)|$, a contradiction.

The subgraph $\bar{G}$ of G induced by $\bar{X}=X-\{a\}$ admits $\bar{H}=H_{\bar{X}}$ as an arborescence forest with $R(\bar{H})\subseteqq R_0$, $S(\bar{H})\subseteqq(S_0-\{a\})\cup\{a_1\}$.

Now, we shall show that $|S(\bar{H})|$ is minimum in $\bar{G}$. Otherwise, there exists in $\bar{G}$ an arborescence forest H' with $R(H')\subseteqq R_0$, $S(H')\subseteqq(S_0-\{a\})\cup\{a_1\}$, $|S(H')|\leqq S|(\bar{H})|-1$. The following cases can happen:

Case 1: $a_1\in S(H')$.

Then $H''=H'+(a_1,a)$ is an arborescence of G which satisfies $R(H'')\subseteqq R_0$, $S(H'')\subseteqq S_0$, $|S(H'')|=|S(H')|\leqq|S(\bar{H})|-1=|S(H)|-1$, a contradiction.

Case 2: $a_1\notin S(H')$, $b\in S(H')$.

Then $H''=H'+(b,a)$ satisfies $S(H'')\subseteqq S_0$, $R(H'')\subseteqq R_0$, $|S(H'')|=|S(\bar{H}')|\leqq|S(H)|-1$, a contradiction.

Case 3: $a_1\notin S(H')$, $b\notin S(H')$.

Then $|S(H')|\leqq|S(\bar{H})|-2$, and $H''=H'+(a_1,a)$ satisfies $R(H'')\subseteqq R_0$, $S(H'')\subseteqq S_0$, $|S(H'')|\leqq|S(H)|$ — a contradiction.

Thus, we have proved the minimality of $|S(\bar{H})|$. So, by the induction hypothesis, there exists in $\bar{G}$ a stable set S which meets every terminal branch of $\bar{H}$. Clearly, S meets also every terminal branch of $H=\bar{H}+(a_1,a)$. ▮

Corollary 1. (Las Vergnas [8]). *Every quasi-strongly connected graph G contains a spanning arborescence with at most $\alpha(G)$ sinks.* (A graph is quasi-strongly connected if for every $x, y\in X$, there exists an ancestor common to x and y.)

Corollary 2. (Linial [10]). *Let $M=\{\mu_1,\mu_2,\ldots,\mu_k\}$ be a path partition of G; either there exists a stable set S which meets each of the μ_i's, or there exists a path partition M' with $S(M')\subseteqq S(M)$, $S(M')\neq S(M)$.*

Corollary 3. (Camion [5]). *Let G be a strongly connected graph such that every pair of vertices is linked with at least one arc ("strong tournament"). Then there exists a Hamilton circuit.*

Proof. Let μ be the largest circuit in G. If μ does not cover the vertex set, every arc going out of μ is the initial arc of a path μ' which comes back into μ (since the contraction of μ gives a graph which is also strongly connected). Let $a \in \mu \cap \mu'$ be the terminal vertex of μ'; the graph obtained from $\mu + \mu'$ by removing the arc $(y, a) \in \mu$ and the arc $(y', a) \in \mu'$ is an arborescence with root a and with sinks y and y'. By the theorem 1, $G_{\mu+\mu'}$ can be covered by an arborescence of root a with only one sink, either y or y'. By adding the arc of G which goes from that sink to a, we form a circuit larger than μ, which is a contradiction. ▮

Corollary 4. *If a graph G has a basis B with* $|B| = \alpha(G)$, *the vertex-set can be covered by* $\alpha(G)$ *disjoint paths all starting from B.* (A basis of $G = (X, U)$ is a set $B \subseteqq X$ such that each vertex is the terminal end of a path starting from B, and no two distinct vertices in B are connected by a path.)

Proof. A basis always exists, by a theorem of König. Consider the sets $B_0 = B_1, B_2, \dots$ where B_i is the set of all vertices which can be reached by a path of length i from B and by no path of length smaller than i. Since B is a basis, $\cup B_i = X$.

Consider a graph H_0 with vertex-set X, obtained by taking for each $x \in B_i$, $i \geqq 1$, one of the arcs of G going from B_{i-1} to x. Clearly H_0 is an arborescence forest with $R(H_0) = B$. By Theorem 1 there exists an arborescence forest H with $R(H) \subseteqq B$, $|S(H)| \leqq \alpha(G)$. However B is a basis, so $R(H) = B$. ▮

When G is a strongly connected graph, one can expect to prove stronger results, that is the existence of better path partitions. Let us mention:

Conjecture 1. (Las Vergnas). *Every strongly connected graph G with* $\alpha(G) \geqq 2$ *has a spanning arborescence H with* $|S(H)| \leqq \alpha(G) - 1$.

Conjecture 2. (Bermond [2]). *The vertices of every strongly connected graph G can be covered by* $\alpha(G)$ *circuits.*

One can cover these two conjectures by another one:

Conjecture 3. *Every strongly connected graph can be covered by a circuit C and* $\alpha(G)$ *disjoint paths having at most this initial end-point in common with C.*

This conjecture has been proved recently for $\alpha(G) = 2$ by C. C. Chen and P. Manalastas [18].

In fact we can apply Theorem 1 to prove the first conjecture for a graph G having the following property:

(P): *G has a circuit which meets every maximum stable set.*

In fact, many graphs satisfy Property (P):

Example 1. (Pósa [12]). A symmetric graph satisfies Property (P).

Let $(y_0, y_1, y_2, \dots, y_k)$ be the longest path issuing from y_0; all the neighbours of y_k are of the type y_i with $0 \leqq i < k$; let i_0 be the smallest index $i \geqq 0$. Then the circuit $\mu=(y_{i_0}, y_{i_0+1}, \ldots, y_k, y_i)$ contains y_k and all its neighbours; therefore every maximal stable set meets the circuit μ. ▮

Example 2. (Meyniel [11]). Not every strongly connected graph satisfies Property (P). Consider a graph G in Figure 1. This graph G has stability number $\alpha(G)=2$, but no circuit meets all the maximum stable sets.

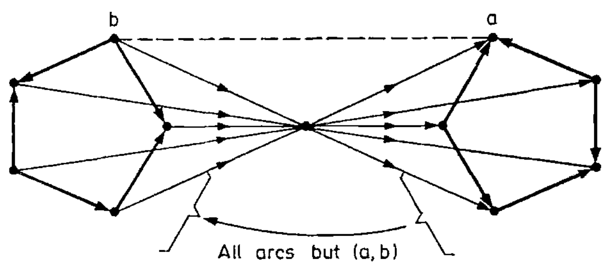

Fig. 1

Theorem 2. *Let $G=(X, U)$ be a strongly connected graph with $\alpha(G)>1$ that satisfies Property (P); then there exists a spanning arborescence H with $|S(H)|\leqq\alpha(G)-1$.*

Proof. Let μ be a circuit which meets every maximum stable set. Let $\tilde{G}$ be the graph obtained from G by contracting μ into a single vertex c. The graph $\tilde{G}$ is strongly connected; so, by Theorem 1, it admits a spanning arborescence $\tilde{H}$ with root c, and $\tilde{G}$ has a stable set S with $|S|=|S(\tilde{H})|$ which meets every terminal branch of $\tilde{H}$.

Clearly, $c\notin S$; so, from the definition of μ, we see that S is not a maximum stable set of G. So, $|S|\leqq\alpha(G)-1$. We can construct in G a spanning arborescence H with arcs of μ and the image of the arcs of $\tilde{H}$, and by removing one of the arcs of μ so that $|S(H)|=|S(\tilde{H})|\leqq\alpha(G)-1$.

This arborescence H fulfills the conditions of the theorem. ▮

Remark. This result shows that the Las Vergnas conjecture is true for symmetric graphs; the same argument shows also that Bermond's conjecture is true for symmetric graphs. However, we can expect better results, and in fact, Amar, Fournier, Germa [1] have conjectured that the vertices of a symmetric graph G can be covered by $[\alpha(G)/2]$ cycles.

By using Theorem 1, we can also extend well-known properties of tournaments to "join" of graphs. The following result has been proved by Las Vergnas [9], and later by Linial [10]:

Proposition 1. *Let $G=(X, U)$ be a graph, let $(A, B, \ldots, K)$ be a partition of X such that $G_A, G_B, \ldots, G_K$ have Hamilton paths $(a_1, a_2, \ldots, a')$, $(b_1, b_2, \ldots, b')$, $\ldots$, $(k_1, k_2, \ldots, k')$, respectively. If every pair of vertices in different classes is joined by at least one arc, then G contains a Hamilton path starting in $\{a_1, b_1, \ldots, k_1\}$ and ending in $\{a', b', \ldots, k'\}$.*

Proof. Clearly it suffices to show the result for a partition (A, B) of X in two classes. In this case, the two Hamilton paths of G_A and G_B constitute an arborescence forest H_0 with $R(H_0) \subset \{a_1, b_1\}$, $S(H_0) \subset \{a', b'\}$. From Theorem 1, this arborescence forest is not minimal, and there exists a single path H with $R(H) \subset \{a_1, b_1\}$, $S(H) \subset \{a', b'\}$. ▮

For strongly connected graphs, the same argument shows the existence of Hamilton circuits. More generally, we get:

Proposition 2. (Las Vergnas, [9]). *Let $G=(X, U)$ be a strongly connected graph; let $(A, B, \ldots, K)$ be a partition of X such that $G_A, G_B, \ldots$ have Hamilton circuits (or are reduced to a singleton). If every pair of vertices in different classes is joined by at least one arc, then for every integer l, $3 \leq l \leq |X|$, the graph G contains a circuit of length l.*

For a tournament G this result was found by Moon (1969). For a complete proof of Proposition 2, see [9].

2. α-diperfect graphs

A directed graph G is *α-diperfect* if for every maximum stable set S, there exists a partition of the vertex-set into paths $\mu_1, \mu_2, \ldots$ such that $|S \cap \mu_i| = 1$ for all i, and if every induced subgraph of G has the same property. Many important classes of graphs are α-diperfect.

Theorem 3. *Every perfect graph is α-diperfect.*

Proof. In a perfect graph G, there exists $k=\alpha(G)$ cliques $C_1, C_2, \ldots, C_k$ which partition the vertex-set, and by Rédei's Theorem, each C_i is spanned by a path μ_i. So, $|S \cap \mu_i| = 1$ $(i=1, 2, \ldots)$. ▮

Theorem 4. *Every symmetric graph is α-diperfect.*

Proof. Let $G=(X, U)$ be a symmetric graph: $(x, y) \in U \Rightarrow (y, x) \in U$. For a maximum stable set S of G consider a graph G' obtained from G by removing the arcs going into S. By the Theorem of Gallai—Milgram, G' has a partition into $k=\alpha(G)=\alpha(G')$ paths $\mu_1, \mu_2, \ldots, \mu_k$. Clearly, the μ_i's are paths of G and satisfy $|S \cap \mu_i| = 1$. ▮

An odd cycle of length $2k+1$ is given by a set of $2k+1$ arcs $u_1, u_2, \ldots, u_{2k+1}$ and determines a sequence of vertices $(x_1, x_2, \ldots, x_{2k+1})$; a *chord* can be either an arc of G joining two non-consecutive x_i's, or an arc of G parallel with one of the u_j's. An odd cycle $(x_1, x_2, \ldots, x_{2k+1})$ is *anti-directed* if
1) its length is >3,
2) the longest path is of length 2,
3) each of the vertices $x_1, x_2, x_3, x_4, x_6, x_8, \ldots, x_{2k}$ is either a *source* or a *sink*,

There are two anti-directed cycles of length 9, shown in Figure 2 but there is only one anti-directed cycle of length 5, or 7.

Proposition 3. *An anti-directed odd cycle without chords is not α-diperfect.*

Proof. If G is an anti-directed odd cycle of length $2k+1$, the set $S=\{x_1, x_4, x_6, \ldots, x_{2k}\}$ is a maximum stable set. If M is a partition into paths which meet S in only one point, then the path of M which contains x_2 is necessarily (x_2, x_1); so the path of M which contains x_3 is necessarily (x_4, x_3); so the path of M which contains x_5 is necessarily (x_5, x_6), etc... Thus, none of the paths of M can be of length 2, which is a contradiction. ▮

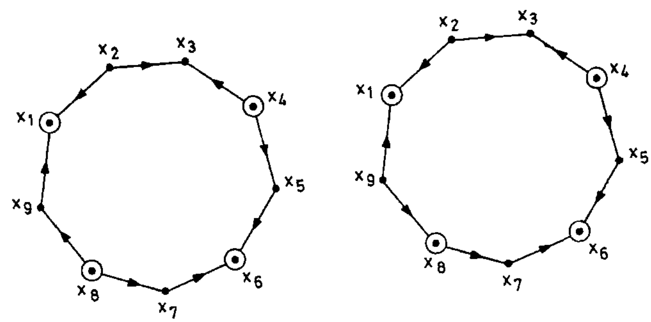

Fig. 2

The graph on Figure 3 is not α-diperfect (because the set $\{a, b\}$ is an optimal stable set which does not have the required property), but it contains anti-directed cycles of length 5. This suggests the following conjectures:

Conjecture 1. A graph G is α-diperfect if and only if G does not contain any anti-directed odd cycle without chords.

Conjecture 2. If every odd cycle (of length >3) has a chord, then G is α-diperfect.

Conjecture 3. If every odd cycle (of length >3) has at least two chords, then G is α-diperfect.

(Each of these conjectures is stronger than the next one.)

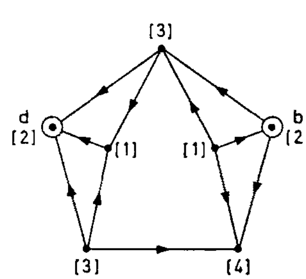

Fig. 3

3. γ-diperfect graphs

Gallai [6] and Roy [13] have proved independently that for a directed graph G, the maximum cardinality of a directed path is at least the chromatic number. A stronger statement can be proved by similar argument:

Theorem 5. *Let k be the maximum number of vertices in a path of G. Then for every path μ with k vertices, there exists a k-coloring $(S_1, S_2, \dots, S_k)$ such that $|S_i \cap \mu| = 1$ for $i = 1, 2, \dots, k$. Furthermore, this k-coloring has the following property: for each $x \in S_i$, there exists an arc going from x to S_{i-1}.*

Proof. Let μ be a path with $|\mu| = k$. Let H be a partial graph of G obtained from μ by adding to μ as many arcs of G as possible without creating a circuit.

Put $\lambda(x) = \max \{|\mu| : \mu$ is a path of H issuing from $x\}$. If (x, y) is an arc of H, then $\lambda(x) > \lambda(y)$ (because H is acyclic).

If (x, y) is an arc of $G - H$, then $\lambda(y) > \lambda(x)$ (because $S + (x, y)$ has a circuit).

So for every arc (x, y) of G, we have $\lambda(x) \neq \lambda(y)$, and therefore $\lambda(x)$ is a coloring function with $\max_x \lambda(x) = k$ colors.

Thus a k-coloring with the required property is defined. ∎

Corollary. *Let $k = \max |\mu|$, and let S be a set of vertices contained in a path μ with $|\mu| = k$. Then the set $A = \{x : x \in X - \mu,\ \Gamma(x) \subset S\} \cup (\mu - S)$ induces a subgraph with chromatic number $\gamma(G_A) \leqq k - |S|$.*

Remark. A generalization of [6] has been obtained by Bondy [3]: If a strongly connected graph G has at least 2 vertices, the longest circuit ("directed cycle") has length $\leqq \gamma(G)$. The Gallai—Roy theorem for a graph H can be proved by using the Bondy theorem for the graph obtained from G by adding a new vertex x_0 that we join in both directions to every vertex of G. Furthermore, it contains also a theorem of Camion [5] which states that a strongly connected tournament has a Hamilton circuit. Remark also that the proof given above does not imply that every graph G has an optimal coloring and a path μ which meets each color exactly once. However, we have:

Proposition 4. *A directed graph G with $\gamma(G) = 3$ has a 3-coloring (S_1, S_2, S_3) and a path $\mu = (a, b, c)$ which meets each color class S_i exactly once.*

Proof. Suppose that G is a graph for which that proposition does not hold true.

Let $(\bar{S}_1, \bar{S}_2, \bar{S}_3)$ be a 3-coloring of G, and let $\mu = (a, b, c)$ be a path of 3 vertices (which exists by the Gallai—Roy Theorem); since μ does not meet the 3 color classes, we have, say, $a, c \in \bar{S}_1$, $b \in \bar{S}_3$. Let S_3 be a maximal stable set which contains $\bar{S}_3$, and put $S_1 = \bar{S}_1 - S_3$, $S_2 = \bar{S}_2 - S_3$. Thus, we have $a \in S_1$, $c \in S_1$, $b \in S_3$.

A vertex x of $G_{S_1 \cup S_2}$ is necessarily a source or a sink. Otherwise, there is a path (y, x, z) with $x \in S_1$, $y, z \in S_2$; since x is adjacent to a vertex $v \in S_3$, there is either a 3-colored path (y, x, v), or a 3-colored path (v, x, z), which is a contradiction.

Let T_1 be the set of non-isolated sources, T_2 be the set of non-isolated sinks and T_0 be the set of isolated vertices in $G_{S_1 \cup S_2}$. Thus, $(T_1, T_2 \cup T_0, S_3)$ is a 3-coloring of G with no 3-colored path and we may assume that

$$a, c \in T_2 \cup T_0, \quad b \in S_3.$$

Every arc between T_2 and S_3 is directed from S_3 (otherwise there would be a 3-colored path). So, $a \in T_0$, and $\mu=(a, b, c)$ meets the 3 colors of the coloring: $(T_1 \cup \{a\}, T_2 \cup T_0 - \{a\}, S_3)$. The contradiction follows. ∎

Remark. Among the applications of the Gallai—Roy Theorem, let us mention:

1. Rédei's Theorem. In every tournament, there exists a path which meets each vertex exactly once.

2. The Chvátal—Komlós Theorem. [16] Let $G=(X, U)$ be a graph, let $U=U_1+U_2+ + \ldots + U_q$ be a partition of the arc set into q classes, let $p_1, p_2, \ldots, p_q$ be integers such that $p_1 p_2 \ldots p_q < \gamma(G)$. Then for some i, the partial graph $G_i=(X, U_i)$ contains a path with more than p_i vertices.

3. The generalized Erdős—Szekeres Theorem. [17] Let $\sigma=(a_1, a_2 \ldots)$ be a sequence of $p_1 p_2 \ldots p_q+1$ distinct integers; let $\varrho_1, \varrho_2, \ldots, \varrho_q$ be binary relations satisfying: for every $i<j$, there exists a relation ϱ_k with $a_i \varrho_k a_j$. Then for some k, there exists a subsequence $\sigma'=(a_{i_1}, a_{i_2} \ldots)$ of σ of length $>p_k$, such that:

$$a_{i_1} \varrho_k a_{i_2}, \; a_{i_2} \varrho_k a_{i_3}, \; \ldots .$$

The proofs are easy and are left to the reader.

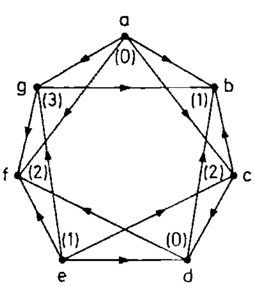

Fig. 4

A directed graph G is *γ-diperfect* if for every optimal coloring $(S_1, S_2, \ldots, S_k)$, there exists a path μ such that $|\mu \cap S_i|=1$ for all i, and if every induced subgraph of G has the same property.

Theorem 6. *Every perfect graph is γ-diperfect.*

Proof. In a perfect graph G there exists a clique C with $|C|=\gamma(G)$, and by Rédei's Theorem, C is spanned by a path μ. So, every optimal coloring $(S_1, S_2, \ldots, S_k)$ satisfies

$$|S_i \cap \mu| = 1 \quad (i = 1, 2, \ldots). \; ∎$$

Theorem 7. *Every symmetric graph is γ-diperfect.*

Proof. Let $(S_1, S_2, \ldots, S_k)$ be an optimal coloring of a symmetric graph G, and let G' be the graph obtained from G by removing the arcs going from S_j to S_i if $j>i$. By the theorem of Gallai—Roy, G' has a path μ with $k=\gamma(G')$ vertices, therefore it meets each of the S_i's exactly once. ∎

The graph in Figure 4, which is isomorphic to the complement $\bar{C}_7$ of a cycle of length 7, is not γ-diperfect, because $\gamma(\bar{C}_7)=4$ and an optimal 4-coloring is represented by the numbers in brackets: one can see that no path of length $\geqq 4$ contains the vertex with color (3).

Also, it is easy to show that the graphs on Figure 2 and on Figure 3 are not γ-diperfect, and that the strong perfect graph conjecture is equivalent to: „A simple graph is γ-diperfect for each orientation of its edges if and only if G is perfect."

Theorem 8. *Let G be an α-diperfect graph on X. Then there exists an optimal coloring $c\colon X\to \{1, 2, \dots, \gamma(G)\}$ such that each vertex x_0 belongs to a path of cardinality $c(x_0)$ that meets each of the colors $1, 2, \dots, c(x_0)$ exactly once.*

Proof. Assume that the result is true for all graphs having chromatic number $<k$; we shall show that it is also true for a graph G with $\gamma(G)=k$.

Let $(S_1, S_2, \dots, S_k)$ be an optimal coloring of G with $S_1\cup S_2\cup\dots\cup S_{k-1}=A$ maximal, and let $x_0\in S_k$. So,

$$\gamma(G_A) = k-1;\ \gamma(G_{A\cup x_0}) = k.$$

By the induction hypothesis, each vertex x in A belongs to a path of G_A with the required properties for some coloring $(S_1', S_2', \dots, S_{k-1}')$ of G_A. Since the graph $G_{A\cup\{x_0'\}}$ is γ-diperfect there exists a path containing x_0 that meets every color-class of the optimal coloring $(S_1', S_2', \dots, S_{k-1}', \{x_0\})$. Therefore each vertex of G belongs to a path with the required property for the optimal coloring $(S_1', S_2', \dots S_{k-1}', S_k)$. ▌

This result is analogous to a theorem of de Werra about perfect graphs [15].

References

[1] D. Amar, I. Fournier and A. Germa, *Private communication,* May 1981.

[2] J. C. Bermond, *Private communication,* February 1978.

[3] J. A. Bondy, Disconnected orientations and a conjecture of Las Vergnas, *J. London Math. Soc.* **2**, 14, (1976), 277—282.

[4] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications,* MacMillan, London, 1972.

[5] P. Camion, Chemins et circuits des graphes complets, *C. R. Acad. Sci. Paris* **249** (1959), 2151—52.

[6] T. Gallai, On directed Paths and Circuits, in: *Theory of Graphs,* (eds. P. Erdős and G. Katona) Academic Press, New York, 1968, 115—118.

[7] T. Gallai and A. N. Milgram, Verallgemeinerung eines Graphentheoretischen Satzes von Rédei, *Acta Sc. Math.* **21** (1960) 181—186.

[8] M. Las Vergnas, Sur les arborescences dans un graphe orienté, *Discrete Math.* **15** (1976), 27—29.

[9] M. Las Vergnas, Sur les circuits dans les sommes complètées de Graphes orientés, Institut de Hautes Etudes de Belgique, *Colloque sur la Théorie des Graphes,* 1973, 231—244.

[10] N. Linial, Covering Digraphs by Paths, *Discr. Math.* **23** (1978), 257—272.

[11] H. Meyniel, *Private communication,* May 1981.

[12] L. Pósa, On the circuits of finite graphs, *Publ. Math. Inst. Hung. Ac.* Sc. **8** (1963) 355—361.

[13] B. Roy, Nombre chromatique et plus longs chemins, *Rev. Fr. Automat. Informat.* **1** (1967), 127—132.

[14] A. SACHE, *La Théorie des Graphes*, Presses Universitaires de France **1554**, Paris 1974.
[15] D. DE WERRA, On the existence of generalized good and equitable edge colorings, *J. of Graph Theory* **5**, n°1 *(to appear)*.
[16] V. CHVÁTAL and J. KOMLÓS, Some Combinatorial Theorems on Monotonity, *Canad. Math. Bull.*, **14** (2), (1971).
[17] P. ERDŐS and G. SZEKERES, A Combinatorial problem in geometry, *Compositio Math.* **2**, (1935), 463—470.
[18] C. C. CHEN and P. MANALASTAS, Every strongly connected digraph of stability 2 has a Hamilton path. *To appear in Discrete Math.*

Claude Berge

Centre de Mathématique Sociale
54 Boulevard Raspail
75270 Paris Cedex 06, France